AP CALCULUS (AB) NAME ______________________

Outline - Chapter 2 - Limits and Continuity Date ________________________

The **limit** is one of the signature concepts of calculus. It separates elementary function analysis from higher mathematics (such as calculus). The process of "finding" a limit is the engine that drives the study of calculus, and allows for the investigation of: 1) quantities and their **instantaneous rates of change**; and 2) quantities and their **accumulation of change**.

In this chapter we define what a limit is, and we determine the value of the limit of a function. The definition is both simple (in what it says) and yet sophisticated (in how it says it). Most of our limits will be determined by one of the following methods: substitution, graphically, numerically, and algebraically (or some combination), depending on the situation. Other methods will be discussed as needed to handle special circumstances.

Limits will also be used to explore the concept of **continuity of functions**; an important characteristic of many functions used to model natural phenomena.

Section 2.1 - Rates of Change and LIMITS

Part A. Limits

1) Evaluating Limits

a. substitution $\lim_{x \to 2} 2x^3 - 7x + 1 =$ _____

b. graphically $\frac{1}{x} * \sin x =$ ______

c. numerically $\lim_{x\to 0} \frac{x+\sin x}{x} =$ ______ **{see handout on LISTS}**

d. algebraically $\lim_{x\to 3} \frac{x^2-9}{x^2-5x+6} =$ ______

Homework 2.1a: page 66 # 7 – 13 odd, 19, 20, 22, 45, 47

2) Properties of Limits – Basic facts that allow us to evaluate the limits of many functions.

a. Theorem 1 – Sum/Difference, Product, Constant Multiple, Quotient, Power Rules, Constant, and Identity Rules (page 61)

Example 3 (page 62) – Application of Limit Properties

b. Theorem 2 – Polynomial and Rational Functions (page 62)

Like many other functions, limits of polynomials and rational functions can be determined by substitution, *where those limit values are defined*!

Example 4 (page 63) – Application of Theorem 2

Example 5 (page 63) – Being Clever!!!

Example 6 (page 63) – Exploring a Limit That Fails to Exist (DNE)

c. Theorem 3 – One-sided and Two-sided Limits (and Limits That Fail to Exist)

Right Hand Limit (RHL) and Left Hand Limit (LHL) and Piecewise Functions

Example 7 (page 64) – Greatest Integer Function

Example 8 (page 64) – Exploring Limits Graphically

d. Theorem 4 – Sandwich Theorem

Certain limits which cannot be determined directly can be determined **indirectly**; one such method is the Sandwich Theorem.

Example 9 (page 65)

Homework 2.1b: page 66 # 16, 37, 39, 40, 42, 49, 51, 55, 61

3) Definition of a Limit (or – It's All Greek to Me!!!)

a. Proving a Limit Exist using the **ε-δ Definition of a LIMIT**:

We say that the $\lim_{x \to c} f(x) = L$ if and only if

given any positive number ε, there is a positive number δ, such that

$| f(x) - L | < \varepsilon$ whenever $0 \leq | x - c | < \delta$.

b. Using the definition of a limit to prove that a limit exists. For our class this will be restricted to linear functions. The last line of the proof will be the numeric relationship between epsilon (ε) and delta (δ).

Example: Prove: $\lim_{x \to 4} 3x - 5 = 7$

Homework 2.1c: Using an ε-δ proof, Prove the following limits:

1) $\lim_{x \to 2} 5x + 3 = 13$ **2)** $\lim_{x \to -3} 2x + 1 = -5$

Part B. Speed of a Particle

1) Average and Instantaneous Speed

The speed of a particle (object) is a specific example of the more general concept of a **rate of change**; speed is the rate at which the *position* of a particle changes with respect to *time*. Therefore any exploration of speed can be extended to the more general notion of a rate of change in any quantity with respect to some other quantity; in many practical applications, the second quantity is very often time, but not always.

To help motivate this concept we will investigate an object in free-fall. According to physics (and of course mathematics), the distance an object travels near the Earth's surface is governed by:

$\mathbf{s = 16t^2}$, where **t** is time measured in seconds, and **s** is distance measured in feet.

a. average speed of a particle during some time interval

- i) Determine the average speed during the first three seconds (i.e. on **[0, 3]**);
- ii) Determine the average speed on **[1, 4]**;
- iii) Determine the average speed on **[2, 3]**;

b. instantaneous speed of a particle (i.e. speed at a specific moment in time)

Determine the instantaneous speed of this particle at $t = 2$ seconds.

- i) Determine the average speed on **[2, 3]**; (see above)
- ii) Determine the average speed on **[2, 2.1]**;

How can I use these average speeds to get a "handle" on the instantaneous speed for this particle at $t = 2$?

c. using the calculator to do the "heavy lifting"

d. using algebra to confirm the numerical investigation above

Homework 2.1d: page 66 # 1, 2, 4, 63, 64

Section 2.2 - LIMITS Involving Infinity

1) Finite LIMITS as x → ±∞ (i.e. limits that exist as x → ±∞)

a. Horizontal Asymptotes

Read opening paragraph on page 70.

Example # 1: Evaluate $\lim_{x\to\infty}\frac{1+2x}{x}$ = _______ **{**graphically, then numerically}

The line y = L is a horizontal asymptote of the graph of the function y = f(x) iff either $\lim_{x\to\infty} f(x) = L$ or $\lim_{x\to-\infty} f(x) = L$.

Does the function f(x) = $\frac{1 + 2x}{x}$ **have any horizontal asymptotes? If yes, write the equation(s).**

Example #2: Determine any horizontal asymptotes for the function:

$$g(x) = \frac{x}{\sqrt{x^2+1}}.$$

Do you notice anything unusual about this function's asymptotic behavior?

b. Sandwich Theorem

Prove: $\lim_{x\to\infty}\frac{\sin x}{x} = 0$

c. Properties/Theorems of LIMITS as x → ∞ (very similar to LIMITS as x → c)

Example: Evaluate: $\lim_{x\to\infty}\frac{5x+\sin x}{x}$.

Homework 2.2a: page 76 # 1, 2, 4, 9, 24,

2) End Behavior Models

End Behavior Models are just that – “simple” functions that model the behavior of given functions at extreme values of x. The models are useful “approximations” for the more complicated given functions as x → ±∞.

a. Right EBM: A function g(x) is a REBM for a function f(x) iff $\lim_{x \to \infty} \frac{f(x)}{g(x)} = 1$.

b. Left EBM: A function g(x) is a REBM for a function f(x) iff $\lim_{x \to -\infty} \frac{f(x)}{g(x)} = 1$.

{Note: If the REBM and LEBM are identical, then it is referred to as the EBM}

Examples: Determine the EBMs for each of the following:

i) $f(x) = \dfrac{2x^5 + x^3 - 4x + 1}{5x^2 - 3x + 7}$

ii) $g(x) = \dfrac{5x^2 - 3x + 7}{2x^5 + x^3 - 4x + 1}$

iii) $m(x) = \dfrac{5x^2 - 3x + 7}{3x^2 - 4x + 1}$

iv) $h(x) = x + e^{-x}$

Homework 2.2b: page # 35, 37, 40, 41, 42

3) Infinite Limits as x → a

Read paragraph on page 72.

a. Vertical Asymptotes

Example: Determine: $\lim_{x \to 1} \dfrac{3x - 1}{x - 1}$

4) Investigating f(x) as x → ±∞ by investigating f(1/x) as x → 0 AND conversely.

a. Substitution – an **indirect method** for evaluating limits.

Example: Determine $\lim_{x \to \infty} \sin \dfrac{1}{x}$.

Homework 2.c: page 76 # 13, 14, 17, 27, 29, 52, 54, 55, 62, 63, 64

AP Review: page 77 # 1 – 4

Quiz #1 – Review Exercises: page 95 # 1, 5, 6, 8, 14, 15 – 20, 27, 33, 34, 35, 42, 52

Section 2.3 – Continuity (Handout)

1) Read Introduction on page 78

2) Example: Handout – solicit answers w/out discussion

3) Definition – Continuity at a Point

   a. interior point – a function, y = f(x), is continuous at an interior point, c ε D_f, iff $\mathbf{\lim_{x \to c} f(x) = f(c)}$.

   b. endpoint – a function, y = f(x), is continuous at a left (right) endpoint iff

   $\mathbf{\lim_{x \to a+} f(x) = f(a)}$ **or** $\mathbf{\lim_{x \to b-} f(x) = f(b)}$

   If a function, f, is not continuous at x = c, we say that **f is discontinuous at x = c**, and that there is a point of discontinuity at x = c.
   {Note: c does not have to be in D_f}

   c. Test for Continuity

   We say that a function f, is continuous at a point x = c, iff

   i) **f(c) exists**
   ii) $\mathbf{\lim_{x \to c} f(x)}$ **exists**

   {Now, reconsider the handout, discuss, and have students figure out the next part}

   iii) $\mathbf{\lim_{x \to c} f(x) = f(c)}$

4) Continuity for Piecewise Defined Functions – page 85 # 48

5) Types of Discontinuity – see page 80

   Removable - criteria “c” definitely fails; criteria “b” definitely passes
   Jump - criteria “b” definitely fails; other criteria could go either way
   Infinite - criteria “b” definitely fails; other criteria could go either way
   Oscillating - criteria “b” definitely fails; other criteria could go either way

Homework 2.3a: page 84 # 2, 3, 7, 10, 11 – 18, 23, 41, 43, 47, 49

6) Continuous Extensions and "Removing Discontinuities" – Exploration #1 (page 81)

7) Continuity over an Interval and Continuous Functions – read page 81

8) Intermediate Value Theorem for Functions – an *Existence Theorem*

If A function f is continuous on a closed interval $[a,b]$, then for all $c \in [a,b]$,

$f(c)$ is between $f(a)$ and $f(b)$.

9) Application – IVT for Functions and the "Existence" of Solutions

Is there any real number which is 3 less than its cube?

Homework 2.3b: page 84 # 25, 27, 29, 31, 45, 56, 57, 58, 59

Section 2.4 - Rates of Change and Tangent Lines

1) Average Rates of Change

   a. Read Introduction – page 87
   b. Example #1: Function – page 87
   c. Example #2: Graphical/Numeric – page 87
   d. Connection: Average Rate of Change and Slope of a Secant Line

2) Secant Lines → Tangent Line

3) Instantaneous Rate of Change and Slope of the Tangent Line

   a. Read page 88 (bottom)
   b. Example #3 – page 89
      i) Slope of a Tangent Line
      ii) Equation of a Tangent Line

Homework 2.4a: page 92 # 1b, 3a, 6b, 7, 10

4) Slope of a Curve - Definition

The **slope of a curve**, y = f(x), at the point P(a, f(a)) is the number

$$m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$, provided the limit exists.

5) Connections Revisited (Restated)

   a. Average Rate of Change = Slope of a Secant Line = Difference Quotient
   b. Instantaneous Rate of Change = Slope of the Tangent Line = LIMIT of the DQ

6) Piecewise Functions and Slope of a Tangent Line

7) Another Example of Slope and Tangents – Example #4 – page 90

8) Determining Equations of Normal Lines to Curves

9) Speed as a Rate of Change – page 91

Homework 2.4b: page 92 # 15, 20, 25, 29, 33*, 36, 37, 38, 39, 40, 41*
Quick Quiz for AP Prep: page 94 # 1 – 4

Quiz #2 – Review Exercises: page 95 # 25, 29, 31, 40, 43, 45, 47, 48, 54, 55