AP CALCULUS (AB) NAME ______________________

Chapter 5 – The Definite Integral Date ________________________

The first part of this course has been driven by the derivative. The derivative represents an instantaneous rate of change of a quantity with respect to an independent variable. Historically the problem situation tied to the derivative has been *motion of a particle*, where the velocity is the derivative of position. This branch is called **Differential Calculus.**

The second "half" of calculus (and this course) investigates how those instantaneous rates of change in a quantity over an interval (usually time) produce an accumulation of some quantity. This branch is called **Integral Calculus.**

Section 5.1 – Estimating with Finite Sums

1) Consider an "Old" Problem: A car travels **150 miles** in **3 hours**.
   a) Set up an solve an equation for the average speed of the car?
   b) Sketch a quick graph of distance v. time (include labeled/titled axes).
   c) What is the connection between our answer and the graph?
   d) What is the connection between our answer and calculus?

2) Now let us do what Leibnitz and Newton must have done; that is to consider the same problem from the other perspective. This means that given the speed of the car, how far will it travel during a certain time interval? At first, we will assume that the car travels at a constant speed, then use what we have learned to extend the idea to varying rates.

   a) A car travels at constant rate of **90 kph** from **2:00pm to 5:00pm**.

      i) Set up and solve an equation to determine how far will the car travel?
      ii) Sketch a quick graph of rate v. time (include labeled/titled axes).
      iii) Is there some connection between our answer and the graph?

   b) What happens if the rate varies over the interval? The velocity of a particle moving along the x-axis is given by:

      $\mathbf{v(t) = 4t + 3, \; t \geq 0}$; where units are: cm/sec and sec.

      i) If the particle starts at the origin (**x = 0**), where is it at **t = 5 seconds**?
      ii) If the particle starts at **x = -3**, where is it at **t = 5 seconds**?

Homework: page 270 # 1, 2

CONTINUE →

c) We next consider the motion of a particle whose velocity is given by a non-linear function.

Example 1 (page 264) – Determining Distance Traveled When Velocity Varies

A particle starts at **$x = 0$** and moves along the x-axis with velocity **$v(t) = t^2$** for time **$t \geq 0$**. Where is the particle at **$t = 3$**?

1) Draw a graph of the function (velocity vs. time) over the given time interval;

2) Partition the time interval [0, 3] into subintervals of length Δt;

3) Draw vertical segments at each endpoint of each subinterval;

Discussion…

4) Choose an "arbitrary" point on each subinterval;

5) Calculate the function value at each of these points;

6) Calculate area approximations for each subinterval; and sum;

****

For purposes of this lab, we will partition the interval [0, 3] into 6 subintervals of equal length

$\Delta t =$ _____

Subintervals: [ ] [ ] [ ] [ ] [ ] [ ]

Arbitrary point:

Function value:
(height)

Area:

Approximate area = __________.

****

Repeat these calculations using different "arbitrary" points.
Use the reverse side of this activity sheet.

Subintervals: [ ] [ ] [ ] [ ] [ ] [ ]

Arbitrary point:

Function value:
(height)

Area:

Approximate area = ___________.

****

Subintervals: [ ] [ ] [ ] [ ] [ ] [ ]

Arbitrary point:

Function value:
(height)

Area:

Approximate area = ___________.

****

Homework: Page 270 # 5, 6, 7, 10, 12

Once you have "mastered" this technique for approximating "area under a curve" and hence, "distance traveled", you will want to take advantage of your graphing calculator's ability to perform some of the less intellectually demanding tasks. Below are the steps to follow:

1) Make certain your rate function is in Y_1;
2) Call up one of the appropriate PROGRAMS (AREA1, INTAREA, or RAM);
3) Input the required field (parameters) values;
4) Record all required values.

NOTE: You are still (and will always be) responsible for being able to perform a RAM by hand. Therefore you will want to maintain an excellent grasp of the entire procedure and concepts for "partitioning an interval" for the purpose of "approximating the accumulative effect of a rate function over an interval."

Section 5.3 – Definite Integrals and Antiderivatives

Recall that a definite integral is defined as the limit of a Riemann sum.

$$\int_a^b f(x)dx = \lim_{n\to\infty}\sum_{k=1}^{n} f(c_k)\Delta x_k \quad \text{on [a, b].}$$

1) How would the value of the definite integral be effected if the limits of integration were reversed?

$$\int_b^a f(x)dx = \quad ? \qquad \text{\{Hint: Think in terms of the definition\}}$$

2) Properties of the Definite Integral

a) Order of Integration (above)
b) [a, a]
c) Constant Multiple
d) Sum/Difference
e) Additivity [a, c], [c, b]
f) Max/Min
g) Domination

Homework 5.3.1: Page 290 # 1, 3, 5, 37, 48

3) Average Value of a Function

a) How does one determine the average of a collection of data?
b) Now how can one use that to determine the *average value of a function*?

Definition – Average (Mean) Value

If f is integrable on [a, b], then its average (mean) value on [a, b] is

$$f_{ave} = \frac{1}{b-a}\int_a^b f(x)dx$$

4) Another Mean Value Theorem?

The Mean Value Theorem for Definite Integrals

If **f** is continuous on **[a, b]** , then at some point **c ε [a, b]**

$$f(c) = \frac{1}{b-a}\int_a^b f(x)dx$$

Homework 5.3.2: Page 290 # 11, 16, 17

Let us pause for a moment (ok, maybe longer) to consider what we have so far accomplished:

1) ***We made a connection between area "under" a velocity curve and displacement (change in position);***
2) ***We extended that notion to include any "accumulation" of change based on any rate function;***
3) ***We used rectangles to approximate area "under" a curve (Riemann Sums);***
4) ***We used the infinite process of limits (of Riemann Sums) to get "exact" values for area "under" a curve;***
5) ***We defined the definite integral to be the limit of a Riemann Sum;***
6) ***We then restated the connection between the definite integral and "area".***

Now let us consider what methods are currently available to us for evaluating a definite integral.

5a) Connecting Differential and Integral Calculus (see Handout)

i. Read page 288;
ii. Exploration 2 (page 289)

5b) Evaluating an Integral Using Antiderivatives

Homework 5.3.3: Page 290 # 19, 21, 22, 23, 27, 29,30, 35

Section 5.4 – Fundamental Theorem of Calculus

1) Fundamental Theorem of Calculus (Part I)

As previously demonstrated (and proven, I hope), we now state it:

Fundamental Theorem of Calculus, Part I
If f is continuous on [a, b], then the function $F(x) = \int_a^x f(t)dt$ has a derivative at every point x ε [a, b], and $\frac{dF}{dx} = \frac{d}{dx}\int_a^x f(t)dt = f(x)$

2) Examples – Applying the Fundamental Theorem of Calculus (Part I)

a) Straight Forward

Homework 5.4.1: Page 302 # 1, 2, 7, 13

b) Chain Rule
c) Reversing the limits of integration

Homework 5.4.2: Page 302 # 9, 13, 17, 20

3) Recovering (Constructing) a Function Given its Derivative and a Value

a) Given: f'(x) = $\sec^2$x and f(π/6) = $\frac{\sqrt{3}}{3}$. Determine f(x).

b) Given: $\frac{dy}{dx} = \tan x$ and $f(3) = 5$. Determine y as a function of x.

Homework 5.4.3: Page 302 # 21, 23

4) The Graph of the Function $y = \int_a^x f(t)dt$

a) Exploration 1 (page 298)

b) The Effect of Changing **a** in $\int_a^x f(t)dt$ (Exploration 2 on Page 299)

5) Fundamental Theorem of Calculus (Part II) – Integral Evaluation Theorem

> If **f** is continuous at every point of **[a, b]**, and
>
> if **F** is any antiderivative of **f** on **[a, b]**, then
>
> $$\int_a^b f(x)dx = F(b) - F(a).$$

a) Proof
b) Example – Evaluating an Integral
c) Area Connection
   i. Analytically
   ii. Using the Graphing Calculator
   iii. Using Graphs to Evaluate Definite Integrals (page 304 # 58)

Homework 5.4.4: Page 302 # 27, 28, 29, 32, 34, 38, 40, 43, 45, 48, 49, 52, 55, 57, 60, 61

Section 5.5 – Trapezoidal Rule

The Trapezoid Rule is a method using trapezoids (as opposed to rectangles) to approximate definite integrals and/or area under a curve.

1) See handout – Water Pollution Problem

2) Area Formula for a Trapezoid: A =

3) Trapezoid Rule – Valid only if all sub-intervals have equal width

Homework 5.5.1: Page 312 # 1, 3, 5, 9

> **Chapter 5 Review Exercises**
>
> **Page 315 # 1, 2, 5, 6, 9, 11, 12bc, 13, 17, 20, 23, 24, 27, 30, 38ab, 39, 40, 45, 48, 54**