AP CALCULUS (AB) NAME ______________________

Chapter 6 – Differential Equations and Modeling Date ________________________

The major theme of this chapter, called *Solving "Initial Value Problems"*, is concerned with "recovering" a function from its derivative and one function value. This process is reflected in the original problem of predicting the future location of a body in motion (such as a planet), knowing its velocity and a previous location. The remaining portions of this unit focus on enhancing our ability to "recover" certain functions by augmenting our "arsenal" of *integration techniques*. Together they are a powerful team for solving a wide variety of real world problems.

Section 6.1 – Slope Fields and Euler's Method

1) Differential Equations – an equation which contains a derivative.

a) Solving a "First-Order" Differential Equation

Consider: $f'(x) = \sec^2 x - 2$. Determine the family of functions, f(x).

This "family" represents the *general solution* of the given differential equation.

b) Solving an "Initial Value Problem"

Consider: $f'(x) = \sec^2 x - 2;\ f(\pi) = 0$. Determine the solution for f(x).

This function, f(x), represents the *particular solution* of the IVP.

c) Solving Another IVP

Consider: $\dfrac{dy}{dx} = \cot x$, where $(\pi/4, 7)$ is a point on the curve.
Determine y as a function of x.

2) Graphing Solutions to Differential Equations and IVP

Given: $\dfrac{dy}{dx} = \cos x$.

a) Graph the family of functions that solve this differential equation;

Note: {-3,-2,-1,0,1,2,3} → L_1 ; $y = \sin x + L_1$

b) Graph the particular function which passes through the point **(π,2)**.

Homework 6.1a: Page 327 # 2, 3, 4, 6, 7, 12, 17, 21, 23, 59

3) Slope Fields – an "approximate" graph for the family of functions that solve a given differential equation; a slope field is obtained by drawing "short" line segments representing the tangents lines to the curves at various points, called lattice points. A slope field is sometimes referred to as a direction field.

a) Constructing a Slope Field by Hand (see Handout)

m = 1 m = -1 m = ½ m = -½ m = 2 m = 2/3

Given: $\frac{dy}{dx} = -x / y$. Construct a slope field at the given lattice points.

b) Constructing a Slope Field Using the Graphing Calculator

i) Construct a SLOPEFIELD for $f'(x) = \cos x$;

ii) Graph the particular solution where $f(\pi) = -2$.

4) Euler's Method – A numerical solution to Differential Equations and IVP's

Given a differential equation and an initial value, determine an approximation for a function value not "too" far from the initial point, by producing a series of approximations for function values "along the way."

Euler's Method requires a differential equation, an initial value, a "step" (dx), and a target function value.

Consider the following IVP: $\frac{dy}{dx} = 2x - 3$ and (1,2)

n	x_n	y_n	dy/dx	dx	y_{n+1}

Homework 6.1b: Page 327 # 33, 41, 43, 47, 49, 53, 62

Section 6.2 – Anti-differentiation by Substitution

1) Indefinite Integral

The family of all antiderivatives of a function, f(x), is the ***indefinite integral of f with respect to x***. The indefinite integral is denoted by $\int f(x)dx$. If F is a function such that F'(x) = f(x), then $\int f(x)dx$ = F(x) + C, where C is the constant of integration.

NOTE: Despite the similarity in notation and the crucial link provided by the Fundamental Theorem of Calculus, there is an important difference between a **definite integral** and an **indefinite integral**. A definite integral is a number (the limit of a sequence of Riemann sums); while an indefinite integral is a family of functions having a common derivative.

2) Evaluating Indefinite Integrals

Evaluating indefinite integrals is akin to solving differential equations.

a) Evaluate: $\int (\frac{1}{x} + \sec x \tan x)dx$

b) See Indefinite Integral Table on Page 332

3) Verify: $\int (\ln u)du = u \ln u - u + C$

4) See Page 333 – Exploration #1 and Example #3abc

5) Substitution – A Technique for Evaluate Indefinite Integrals

{NOTE: u-substitution is easily recognized as a method to make the Anti-differentiation of composition easier, but it can be used in other situations as well}

Homework 6.2a: Page 337 # 3, 4, 5, 9, 12, 17, 18, 21, 22, 25, 27, 29, 33, 39, 40, 41

Section 6.3 – Antidifferentiation by Parts

Do Now: Evaluate: $\int x * \cos x dx$

1) Recall the *product rule for derivatives*: $\dfrac{d[f(x)g(x)]}{dx} = f(x)g'(x) + g(x)f'(x)$

Let u = f(x) and v = g(x); $\dfrac{d[uv]}{dx} = u * \dfrac{dv}{dx} + v * \dfrac{du}{dx}$

2) Now take the equation above and integrate both sides wrt x.

$$\int \frac{d[uv]}{dx} dx = \int u * \frac{dv}{dx} dx + \int v * \frac{du}{dx} dx$$

What does this yield?

Rearrange:

{NOTE: This formula, known as *Integration by Parts* is a method for anti-differentiating a product (in essence it is a product rule for integrals). It can also be used for integrands that do not initially present themselves as products as well.}

3) Reconsider the above example and evaluate: $\int x * \cos x dx$

4) What Happens if You Choose the Parts Incorrectly? {LIPET}

Homework 6.3a: Page 346 # 5, 9, 10, 11, 30,

5) Applications

a) Determine the area between: $y = xe^{-x}$ and the x-axis on [0, 3].

b) Evaluate: $\int x^2 e^x dx$

c) Evaluate: $\int e^x \cos x dx$

6) Let’s take another look at (5b)

7) Evaluate: $\int \ln x dx$

Homework 6.3b: Page 346 # 15, 17, 23, 25, 35, 42, 43

Section 6.4 – Exponential Growth and Decay

1) Separable Differential Equation – a differential equation of the form ***dy/dx = f(y)*g(x)*** is called separable. A separable differential equation can be solved by "separating" the variables as follows: $\frac{1}{f(y)}dy = g(x)dx$. The solution is found by anti-differentiating both sides with respect to the separate variables.

   a) Example: **dy/dx = (xy)^2** ; when **x =1 , y = 1**.

   b) Example: **dy/dx = -x/y** and **y = 3,** when **x = 4**.

Note: 1) Since quotients can be expressed as products, look to use separation of variables if the derivative is expressed as a product or quotient of two functions (one involving the independent variable; the other involving the dependent variable).

2) Look to use *separation of variables* whenever the derivative is expressed in terms of the dependent variable (it may also involve the independent variable).

Homework 6.4a: Page 357 # 1, 4, 7, 8, 10

2) Exponential Change (Growth and Decay)

   Exponential functions are used to model quantities in which *the rate of change (growth/decay) is proportional to the amount present.* To understand why this is so, we will express the rate of growth as a differential equation; and apply the method of solving a separable differential equation.

   Let **y** be a quantity that varies over time, **t**. The rate of change of **y** is thus proportional to **y** itself! We will express this relationship using a differential equation, and then solve for **y**.

Homework 6.4b: Page 357 # 11, 13

3) Applications and Using Exponential Models

   a) Growth: Compound Interest – Page 352 Example # 2

   b) Decay: Page 359 # 40

   c) Doubling Time and Half-life

Homework 6.4c: Page 357 # 19d, 21, 23, 25, 35, 37, 38, 40

4) Continuous Compounding Rate, Growth Factor, and Effective Rate

5) Differences Between Growth and Decay

6) The Rule of 70

Section 6.5 – Logistic Growth

In the previous section (and in previous years) you explored exponential growth/decay. Exponential functions are used to model real-world phenomena where the rate of growth/decay is proportional to the amount of a quantity present; some areas in which exponential models are used to predict future values are: populations, investments, radioactive decay, etc. However, as you are aware from other coursework and experience, certain quantities do not grow unchecked; and other quantities do not decline to zero. For these situations a different type of function is required, called a Logistic Model. Before exploring the calculus of logistic functions, we introduce one more technique for integration, based on an algebraic technique from pre-calculus.

1) Before moving on, let's take some time to evaluate the indefinite integrals below:

a) $\int \frac{1}{x} dx =$ b) $\int \frac{1}{x-3} dx =$ c) $\int \frac{1}{2x+1} dx =$

d) $\int \frac{x+1}{x^2+2x} dx =$ e) $\int \frac{4}{x^2+36} dx =$ f) $\int \frac{1}{x^2+2x} dx =$

2) It is this last integral which points out the need for an additional integration technique.

a) Recall the following:

i. x^2+2x is a quadratic, factorable into linear factors; what are they?

ii. $\frac{1}{x^2+2x}$ is a fraction possibly resulting from the addition or multiplication of two other fractions. Which one would make for an easier integration?

b) $\frac{1}{x^2+2x} =$

c) Hence: $\int \frac{1}{x^2+2x} =$

Homework 6.5a: Page 369 # 19, 9, 5, 43

NOTE: 1) You are only responsible for denominators can be factored into linear factors;
2) The Heaviside Method gives us a shortcut for determining the numerators without having to show the steps of solving simultaneous equations.

3) The Logistic Differential Equation

See class notes!!!

Homework 6.5b: Page 369 # 23, 27, 33
Page 372 # 3, 4

Chapter 6 Review Exercises
Page 372 # 1 – 3, 5 – 8, 10, 13, 15 – 17, 20, 22, 25, 32, 35, 38, 43, 48, 49, 54, 57, 60, 63b, 69